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Random packings of graphs

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Dedicated to the Memory of Sergio Ruiz (1952–1991)

Abstract

A graph G is said to be packable by the graph F if its edges can be partitioned into copies of F. It is called randomly packable if what remains after deletion of the edges of a proper subgraph that is F-packable is also F-packable. We establish some results on graphs that are randomly packable by matchings, paths, complete graphs, and others.

0. Introduction

The question that we consider is the following: Given a graph F, which graphs G have the property that every collection of edge-disjoint copies of F in G can be extended to cover all of G?

More formally, we make the following definitions: For graphs F and G, a collection of edge-disjoint subgraphs of G that are isomorphic to F is called a *partial F-packing* of G, and it is a *full F-packing* if every edge of G is in one of the copies of F. In addition, we say that G is *F-packable* if it has a full F-packing and it is *randomly F-packable* if every partial F-packing can be extended to a full one.

One example of random packability is that $K_{2,2r}$ is randomly C_4-packable. In contrast, the octahedron, although K_3-packable, is not randomly so, since a set of two vertex-disjoint triangles is not extendible to a larger packing, as we indicate in Fig. 1.

The concept was introduced by Ruiz [6] under the name of 'randomly decomposable graphs,' and he found all randomly F-packable graphs for the two cases in which

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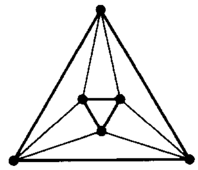

Fig. 1.

F has two edges. The connected result was extended to all stars by Barrientos et al. [2]. We present that result in Section 1. In Section 2, we obtain an extension of Ruiz's other result to larger sets of independent edges. In later sections, we consider complete graphs, paths, and cycles and other graphs.

The following observations are elementary, but crucial to our work. First, if G is randomly F-packable and H is an F-packable subgraph of G, then H itself must be randomly F-packable. (Since H is F-packable, $G-H$ must also be, and its packings must be randomly extendible.) Consequently, if a graph G contains a subgraph that is F-packable but not randomly so, G itself cannot be randomly F-packable. For convenience, we call such a subgraph *F-forbidden* (or just *forbidden* if F is clear by context). We formalize these statements in the following lemma.

Basic packability lemma. *Let G be an F-packable graph and H an F-packable subgraph of G.*

(i) *If G is randomly F-packable, so is H.*

(ii) *If H is F-forbidden, then G is not randomly F-packable.*

Note: Because of this hereditary nature of packability, it follows that for each F, there is a family of minimal F-forbidden graphs. Partial results on determining some of these families are implicit in our proofs, but we will not pursue this topic here. Caro et al. [4] have studied it from an algorithmic perspective.

Some points about notation and terminology. For obvious reasons, we assume that none of the graphs in this paper have any isolated vertices. Even though it may give the statements of theorems an inconsistent appearance, it is simpler to state F-packability results only for connected graphs G if F itself is connected. Of course, if F is disconnected, then G is allowed to be any graph (without isolated vertices).

If H is a subgraph of G, we let $G-H$ denote the graph left after we delete from G the edges of H and any resulting isolated vertices. We also let $G+H$ denote the union of two disjoint graphs (so that $2G=G+G$), and save $\cup$ for the usual set-union operation where the graphs may overlap. The graph obtained from two n-cycles by identifying one vertex from each will be denoted $C_n \cdot C_n$. Following the usual custom, the path of length n will be denoted P_{n+1} but called an *n-path.*

1. Stars

Caro and Schönheim [5] proved that a connected graph is $K_{1,2}$-packable if and only if it has an even number of edges and (as noted in the Introduction), Ruiz [6] determined which ones are randomly packable.

Theorem 1.1. *The only connected randomly $K_{1,2}$-packable graphs are the cycle C_4 and the stars $K_{1,2t}$.*

This result was later generalized to arbitrary stars by Barrientos et al. [2].

Theorem 1.2. *For $r \geqslant 2$, a connected graph G is randomly $K_{1,r}$-packable if and only if it is $K_{r,r}$ or it is bipartite with all degrees in one partite set being multiples of r and all degrees in the other set being less than r.*

2. Matchings

A set of independent edges is called a *matching*, and a matching with t edges will be denoted by M_t. Several results related to packing with matchings appear in the literature. One of these, due to Alon [1], determines the M_t-packable graphs, given that the number of edges is sufficiently large.

Theorem 2.1. *Let G be a graph with q edges and maximum degree d. For $q > 8t^2/3 - 2t$, G is M_t-packable if and only if $t \mid q$ and $q \geqslant td$.*

(We note that Sumner [7] considered a different form of random packing, in which every partial matching is extendible to a full one.)

Theorem 2.2. *The only connected graphs with the property that every partial matching can be extended to a full matching are K_{2t} and $K_{t,t}$.*

Turning to random packability, we next give Ruiz's [6] characterization of randomly M_2-packable graphs.

Theorem 2.3. *A graph is randomly M_2-packable if and only if it is one of the following: $C_4, K_4, 2K_3, K_3 + K_{1,3}, 2K_{1,n}$, or $2nK_2$ $(n \geqslant 1)$.*

We will use Alon's result to characterize those M_t-packable graphs with sufficiently many edges. We also make use of the following elementary lemma, similar to one of Bialostocki and Roditty [3].

Lemma 2.4. *Let r and q be positive integers with $q > 2r^3 - r^2$. In any graph with q edges, at most r vertices have degree q/r or greater.*

(We observe that the proof can be readily adapted to any real number $r \geqslant 1$, but we need it only in the integer case.)

Proof. The result is clearly true for $r = 1$, so we assume that $r > 1$. Let S be the set of vertices of degree at least q/r, and let $s = |S|$. We suppose that $s > r$. Let x be the number of edges with both ends in S and let y be the number with just one end there. Then $x + y \leqslant q$ and $2x + y \geqslant sq/r$, so $x \geqslant q((s/r) - 1)$. This has two consequences. On the one hand, since $s > r$, it implies that $q \leqslant (rx/(s-r)) \leqslant rx$, and on the other, since $x \leqslant q$, it implies that $s \leqslant 2r$. Consequently, since $x \leqslant \binom{s}{2}$, we have $q \leqslant r \binom{s}{2} \leqslant 2r^3 - r^2$, which contradicts the hypothesis. □

The following result gives those randomly t-matching-packable graphs with sufficiently many edges.

Theorem 2.5. *For a given integer $t \geqslant 2$, a graph with at least $2t^3 - t^2$ edges is randomly M_t-packable if and only if it is isomorphic to tH, where H is either nK_2 or $K_{1,n}$ for some $n \geqslant 1$.*

Proof. Clearly, the given graphs tH are randomly M_t-packable. For the converse, assume that G is a graph with nt edges that is randomly M_t-packable, with $n \geqslant 2t^2 - t$. By the basic lemma, if J is a subgraph of G with t edges, not all independent, then $G - J$ is not M_t-packable. Consequently, by Theorem 2.1, $\Delta(G) < n$, and for such a subgraph J, $\Delta(G - J) > n - 1$. It follows that if $\Delta(G) < n$, then every set of t edges is independent, and so $G = ntK_2$. Therefore, we assume that $\Delta(G) = n$ and let S denote the set of vertices of degree n. By Lemma 2.4 (with $r = t$), $|S| \leqslant t$. Let J be any subgraph with t edges covering S, and let $\Delta(G - J) = d$. Then $d \leqslant n - 1$, and the number $nt - t$ of edges in $G - J$ is at least td and at least $8t^2/3 - 2t$, so by Theorem 2.1, $G - J$ is M_t-packable. Consequently, the edges of J must be independent, S must have order t, and no two vertices in S can have a common neighbor. By counting edges, we conclude that G is $tK_{1,n}$. □

3. Complete graphs

Graphs that are randomly packable by complete graphs of a given order are very simple to describe.

Theorem 3.1. *A graph G is randomly K_n-packable if and only if every edge lies in precisely one copy of K_n in G.*

Proof. Clearly, the graphs described in the theorem are randomly K_n-packable. Let G be randomly K_n-packable, and suppose that some edge lies in two copies of K_n, F and H. Consider a K_n-packing of G that contains F, and let J be the union of those copies $F=F_1, F_2, \ldots, F_m$ of K_n in that packing that contain edges of H. By the basic lemma, since J is K_n-packable, $J-H$ must be K_n-packable. Let v be a vertex in J but not in H, and let r be the number of F_i that contain v, so that v has degree $r(n-1)$ in J. Since each F_i has at least two vertices of H, there must be at least $2r$ edges between v and H. Each of these edges must lie in a different copy of K_n in a packing of $J-H$, so that the degree of v in J must be at least $2r(n-1)$, an impossibility. □

4. Paths

A general result stating which graphs are randomly n-path-packable seems difficult to obtain, and we have complete results only for $n \leqslant 6$, The 2-path is of course the same as $K_{1,2}$, and so Ruiz's theorem (Theorem 1.1) covers that case. Before presenting our results for 3-, 4-, and 5-paths, we establish two lemmas of some interest in their own right. The first says that, except for certain subdivided stars, randomly n-path-packable graphs must have cycles, and the second that such cycles must be relatively short. For convenience, we let $S_{2k}^{(r)}$ denote the graph obtained from r paths of length $2k$ by identifying their center vertices (see Fig. 2 for $S_4^{(3)}$).

Lemma 4.1. *For $n>1$, the only randomly n-path-packable trees are P_{n+1} itself and, when n is even, the subdivided stars $S_n^{(r)}$.*

Proof. We first observe that if a tree T has a path of length greater than n, then it cannot be randomly P_{n+1}-packable. To see this, simply consider a longest path Q in T and begin packing T with an n-path at a penultimate vertex of Q.

Next, we note that if the tree T is the union of two edge-disjoint n-paths, then it contains an $(n+1)$-path unless n is even and the two paths have their center vertices in common. Thus, these are the only trees with $2n$ edges that are randomly P_{n+1}-packable. The full result for trees with more edges follows at once. □

Lemma 4.2. *Let $n \geqslant 4$, and let G be a connected randomly P_{n+1}-packable graph. If G has a cycle of length at least $n+1$, then G is C_{2n}.*

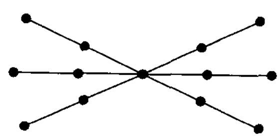

Fig. 2.

Proof. Assume that G is a randomly P_{n+1}-packable graph having a cycle C of length $s \geqslant n+1$. Since P_{2n+1} is P_{n+1}-forbidden, $s \leqslant 2n$. If $s=2n$, then G must equal C_{2n} since a $2n$-cycle together with an incident edge always contains a forbidden graph. Furthermore, the union of C_{2n-1} and an incident edge is also forbidden, so we assume that $s \leqslant 2n-2$.

Let $C=v_0v_1 \dots v_{s-1}v_0$ and let $Q_i := v_iv_{i+1} \cdots v_{i+n}$ (subscripts modulo s) for $i=0,1,\dots,s-n$. Let Q_0' denote the n-path containing the edge v_nv_{n+1} in a packing of $G-Q_0$. Let H be the union of Q_0 and Q_0'. Note that if the n-path Q_i is contained in H, then the graph $Q_i' := H - Q_i$ is also an n-path since H is randomly P_{n+1}-packable. We now consider two cases.

Case 1: H contains C. Being the union of two paths, H has at most four vertices of odd degree. For each i, the n-path Q_i' contains the other portion of C, $v_{i+n}v_{i+n+1} \cdots v_i$, which has length less than n. Consequently, each Q_i has at least one end of degree 3 in H. Since there are at least five such paths, Q_i, H must have more than two, and hence four, vertices of odd degree. Furthermore, at least one Q_i must have one end of even degree, so the corresponding Q_i' has three vertices of odd degree, an impossibility.

Case 2: Some edge of C is not in H. First observe that since H is randomly P_{n+1}-packable, if R is any n-path in H, then $H-R$ must also be an n-path. We note that one consequence of this is that we may assume that the edge $v_{s-1}v_0$ is not in H, since if it were, we could just back Q_0 up until that situation is obtained.

We next establish the following claim (which we will utilize several times): *Let $W=w_0w_1 \cdots w_m$ be a path in H with $m>n$. Let $R=w_0w_1 \cdots w_n$, $S=H-R$, $Z=W-R$, and $A=w_0w_1 \cdots w_{m-n}$. Also, let $R'=R+Z-A$, whence $S'=S+A-Z$ is also an n-path. Consequently, either*

(i) *the ends of S are the ends of A, or*

(ii) *one end of S is an end of A and the other an end of Z.*

With $P=v_0v_1 \cdots v_n$, let r be such that all edges of $Y=v_nv_{n+1} \cdots v_r$ are in H (we know $r>n$) but that v_rv_{r+1} is not, and let k be such that no edge of $X=v_kv_{k+1} \cdots v_0$ is in H but that the edge $v_{k-1}v_k$ is (possibly $k=r$). In addition, let $B=v_0v_1 \cdots v_{r-n}$. Now let P, Q, Y, and B play the roles of R, S, Z, and A in the claim. It follows that the ends of Q are therefore either (i) v_0 and v_{r-n} (the ends of B) or (ii) one is v_0 or v_{r-n} and the other v_n or v_r.

By keeping Q and C fixed (but changing H), we can also let the path $v_{n+k}v_{n+k-1} \cdots v_k$ in $G-Q$ play the role of R, Q that of S, and the edges v_kv_{k-1} and $v_{n+k}v_{n+k-1}$ those of Z and A, respectively. Hence, at least one end of Q must be v_{n+k} or v_{n+k-1}, and if not both, the other must be v_k or v_{k-1}. This eliminates the possibility that v_0 is an end of Q, and hence v_{r-n} must be, as well as either v_n or v_r. Consequently, by our choices of r and k, it must be the case that $k=r$ and v_r is an end of Q.

We can also apply the claim to P and Q with Z being v_nv_{n+1} to deduce that v_1 is an end of Q and that $r=n+1$. This means that v_{n+k} or v_{n+k-1} is v_1, and hence $s=2n$ or $2n-1$. This contradicts the condition that $s \leqslant 2n-2$, and completes the proof. □

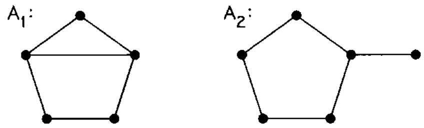

Fig. 3.

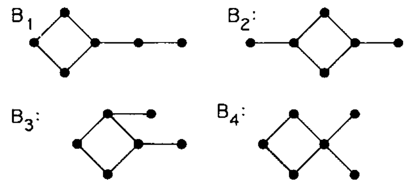

Fig. 4.

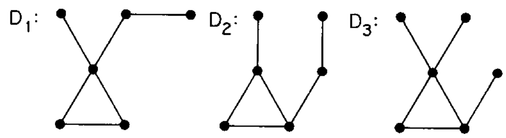

Fig. 5.

In our next three theorems, we determine the families of randomly n-path-packable graphs for $n=3,4$, and 5. We begin the length-3 case by noting that each of the graphs in Figs. 3–5 is P_4-forbidden.

Theorem 4.3. *The only connected randomly P_4-packable graphs are P_4, K_4, $K_{2,3}$, C_6, and $C_3 \cdot C_3$.*

Proof. Each of the five graphs is readily seen to be randomly P_4-packable. For the converse, let G be a connected randomly P_4-packable graph. By Lemma 4.1, we may assume that G has a cycle, and we let n be the length of a longest one. We now consider several cases.

Case 1: $n \geqslant 7$. This is impossible since such a cycle contains the forbidden graph P_7.

Case 2: $n=6$. Since C_6 itself is randomly P_4-packable, we consider the case where G has additional edges. One such edge must have one or both ends in C_6, and the result (there are three possibilities) always contains a forbidden graph.

Case 3: $n=5$. A 5-cycle must have an edge incident with it, and so G must contain one of the graphs in Fig. 3.

Case 4: $n=4$. It can be readily verified that if G has order 4, then it must be K_4, and if order 5, $K_{2,3}$. So we suppose that the order of G is at least 6. Let $C=uvwxu$ be a 4-cycle. Then there must be a vertex y adjacent to a vertex on C, say x, and another vertex z adjacent to one of the other five. Up to isomorphism, there are just the four possibilities shown in Fig. 4, and each of these is forbidden.

Case 5: G has only 3-cycles. Let u, v, and w form a triangle, and let x be another vertex adjacent to v. Since $wuvx$ is a 3-path, the edge vw must be on a 3-path, that is edge-disjoint from this and which must therefore take one of these forms: *wvyz*, *vwyz*, or *yvwz*, for some vertices y and z. If the six vertices are all distinct, then there are just the three possibilities shown in Fig. 5 and each is readily seen to be P_4-forbidden. Therefore, either y or z must coincide with u, v, w, or x. When we consider the various possibilities, we see that a 4-cycle appears in every case but one, that one yielding $C_3 \cdot C_3$. Now suppose that G contains $C_3 \cdot C_3$ as a proper subgraph. It cannot have another edge without another vertex since there would be a longer cycle. But if it has another vertex, then it must contain graph D_1, which it cannot. Thus, G can only be $C_3 \cdot C_3$ and the proof is complete. □

Theorem 4.4. *The only connected randomly P_5-packable graphs are $P_5, K_{2,4}$, $C_4 \cdot C_4$, C_8, and $S_4^{(k)}$ for $k \geqslant 2$.*

Proof. Each of the given graphs is clearly randomly P_5-packable. Using the lemmas and an exhaustive search, one can verify that this list includes all such connected graphs with eight edges. (We omit the details.)

Now assume that G is a connected randomly P_5-packable graph with 12 edges. By the lemmas, we may assume that G has a cycle and the maximum cycle length is 4. Therefore, some pair of 4-paths must form either $K_{2,4}$ or $C_4 \cdot C_4$. Suppose it is $K_{2,4}$. Then G cannot have another vertex since it would contain one of the graphs shown in Fig. 6, both of which contain a P_5-forbidden subgraph. But one cannot add a P_5 to $K_{2,4}$ without forming a 5-cycle. Similarly, if G contains $C_4 \cdot C_4$ and has another vertex, then it contains a P_5-forbidden subgraph as indicated in Fig. 7. But if a path of length 4 is added to $C_4 \cdot C_4$, some edge must join vertices of degree 2 in different cycles, resulting in a longer cycle. Hence, the only randomly P_5-packable graphs with 12 edges, or more, are subdivided stars. □

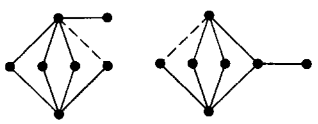

Fig. 6.

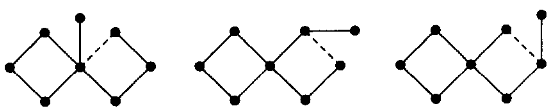

Fig. 7.

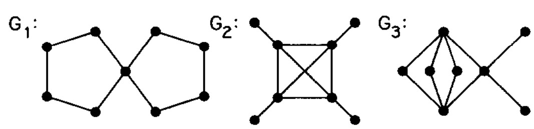

Fig. 8.

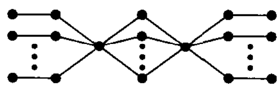

Fig. 9.

Theorem 4.5. *The only connected randomly P_6-packable graphs are P_6, C_{10}, and the three graphs in Fig.* 8.

Proof. It is a routine matter to verify that these five graphs are randomly P_6-packable and that the last four are the only connected graphs with ten edges that are. (We omit the details.)

Assume that G is a connected randomly P_6-packable graph with 15 edges. It follows from the basic lemma that removing the edges of any 5-path must result in either G_1, G_2, or G_3. Note that none of them has a vertex of degree 3.

First, suppose G contains G_1. If there is another vertex in G, then there would be a subgraph consisting of a 5-path and a 5-cycle joined at an internal vertex of the path. Since such a graph is P_6-forbidden, G can have only nine vertices. One end vertex of the path $G-G_1$ must have degree 3 in G. However, G contains a 5-path S avoiding that vertex. But then $G-S$ is not randomly P_6-packable, and this is a contradiction.

Therefore, the union of any two edge-disjoint 5-paths in G must form G_2 or G_3. Let P, Q, and R constitute a P_5-packing of G, with $P = v_0 v_1 \cdots v_5$. Then whether $P \cup Q$ is G_2 or G_3, Q must contain edge $v_1 v_4$. Since the same can be said for R, there cannot exist three such paths. Hence, there is no connected randomly P_6-packable graph with 15 or more edges. $\square$

Y. Caro and J. Rojas (personal communication) have observed that, for larger n (both odd and even), paths can be amalgamated at places other than their central vertices to get other infinite families of randomly P_n-packable graphs. For example, Fig. 9 shows such a family for P_7. A significant feature of these graphs is that the only paths of length n join two end vertices. The same is true of the graphs obtained by extending the legs of G_2 in Fig. 8.

5. Cycles and other graphs

The most natural connected families of graphs to study beyond those already considered would be cycles and complete bipartite graphs. Thus, C_4 (being also $K_{2,2}$) is the next graph to look at, and things seem to be quite complicated here. Clearly, the complete bipartite graphs $K_{2,2r}$ are all randomly C_4-packable, and one can trivially generate others by taking unions in such a way that no new 4-cycles are formed (cf. the result on complete graphs). Others can be generated from the family of $K_{2,2r}$'s by

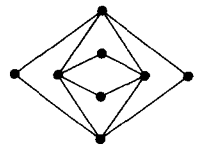

Fig. 10.

repeatedly attaching such a graph between two vertices of degree 2 (see Fig. 10). However, $K_{4,4}$ is also randomly C_4-packable, and it has no vertex of degree 2. Determining the family of randomly C_4-packable graphs has eluded us thus far, although we have made progress in determining the minimal forbidden subgraphs. This will be developed in future work.

One might also consider randomly F-packable graphs for other disconnected graphs than the matchings, and here we will mention just two, $F=P_2+P_3$ and $F=K_2+K_3$, because the results are quite different. For the first (besides itself), there is an infinite family of randomly packable graphs ($K_{1,2r}+rK_2$), as well as seven others with six edges. However, in the second case, there are no randomly F-packable graphs other than F itself. On the surface, these results are somewhat surprising when one considers the similarities of the two graphs.

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